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On Levitin–Polyak well-posedness and stability in set optimization

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Abstract
In this paper, Levitin–Polyak (in short LP) well-posedness in the set and scalar sense are defined for a set optimization problem and a relationship between them is found. Necessary and sufficiency criteria for the LP well-posedness in the set sense are established. Some characterizations in terms of Hausdorff upper semicontinuity and closedness of approximate solution maps for the LP well-posedness have been obtained. Further, a sequence of solution sets of scalar problems is shown to converge in the Painlevé–Kuratowski sense to the minimal solution sets of the set optimization problem. Finally, the perturbations of the ordering cone and the feasible set of the set optimization problem are considered and the convergence of its weak minimal and minimal solution sets in terms of Painlevé–Kuratowski convergence is discussed.

Keywords Set optimization · Levitin–Polyak well-posedness · Painlevé–Kuratowski convergence · Stability

Mathematics Subject Classification 49J53 · 49K40 · 90C31 · 90C48

1 Introduction

Set optimization is an important generalization of vector optimization problems. It has wide applications in several areas such as mathematical finance, game theory, duality principles, multiobjective optimization, gap functions for vector variational inequalities, fuzzy programming, welfare economics (see [2,8,15] and the references therein). To find the optimal solution of set optimization problem, two well known

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solution criteria are considered namely vector criterion and set criterion (see [13,15,23]). Vector criterion is a generalization of solutions of vector optimization problem whereas the set criterion is based on minimal elements obtained using quasiorders on image sets of the objective function map. Set criterion have been used throughout this paper.

Well-posedness of an optimization problem means that the variables whose objective function values are close to the optimal value ultimately lie close to solution of the problem. It plays an important role in the study of sensitivity and stability of optimization problems. Tykhonov [25] was the first to introduce well-posedness for scalar optimization problems which was based on the idea of minimizing sequences converging to the unique solution of the problem. In this approach, the minimizing sequences are required to stay in the feasible region of the problem which need not be always true. To overcome this limitation, Levitin and Polyak [19] extended this concept of well-posedness where the minimizing sequences could also lie outside the feasible set but required their distance from it to approach zero, this well-posedness is termed as Levitin Polyak well-posedness. In literature, these concepts have further been extended for vector optimization problems by many authors (see [3,21,24] and the references therein). Chatterjee and Lalitha [3] studied LP well-posedness for a vector optimization problem in both the vectorial and scalar sense and obtained a link between them. They established some sufficient conditions for LP well-posedness and discussed some stability aspects of the vector optimization problem.

Well-posedness for set optimization problems have also been explored extensively and various notions of well-posedness have been defined and studied by many researchers (see [4–6,9,10,16,28] and the references therein). This study was initiated by Zhang et al. [28] where various types of well-posedness in set optimization have been defined and certain characterizations for them have been obtained. Later, Crespi et al. [5] proposed a global notion of well-posedness which generalized one of the notions of well-posedness given by Zhang et al. [28]. Khoshkhabar [16] extended the concept of well-posedness given in [28] by introducing the concept of generalized LP well-posedness and obtained few characterizations of this newly defined well-posedness in terms of closedness and upper semicontinuity of an approximate solution mapping. Vui et al. [26] introduced different kinds of LP well-posedness for set optimization problems by considering various types of set order relations. They investigated some necessary and sufficient conditions for these LP well-posedness and established characterizations for these concepts via Kuratowski measure of noncompactness. It has been observed that these well-posedness are suitable only for solid optimization problems as the ordering cone is required to have nonempty interior. This is a limitation because there exist optimization problems in literature where the ordering cone has an empty interior. Such problems are known as nonsolid optimization problems. To overcome this shortcoming, recently Gupta and Srivastava [6] introduced different types of well-posedness for set optimization problems which can be used for both solid and nonsolid optimization problems and have obtained their necessary and sufficiency criteria. Using generalized oriented distance function [27], they established some scalar equivalences of these well-posedness.

The study of stability of an optimization problem has a significant role in optimization theory. Many authors have studied stability aspects of vector optimization problems by taking into account the perturbations of the ordering cone, the feasible set and the objective function (see [18,20,22] and the references therein). Luc et al. [22] investigated the convergence of the solution sets of perturbed vector optimization problems in the sense of Painlevé–Kuratowski convergence when both the ordering cone and the feasible set are perturbed. Li et al. [20] established some stability results of a vector optimization problem by considering the perturbation of the ordering cone, the feasible set and the objective function. The study of stability analysis pertaining to perturbations of the data has been investigated by some authors in the area of set optimization (see [7,10,14]). Gutiérrez [7] considered the perturbation of the feasible set and established external and internal stability of the solution sets of a set optimization problem by using the notion of Hausdorff and Painlevé-Kuratowski set convergence. Later, Karuna and Lalitha [14] improved some results of [7] by establishing external and internal stability in both the image space as well as in the given space.

Motivated by the above work, in this paper we introduce a type of LP well-posedness in the set sense and scalar sense for a set optimization problem and a link is established between them. Some necessary and sufficient conditions have been obtained for these LP well-posedness by invoking Painlevé-Kuratowski set convergence and u-domination property, respectively. A few characterizations of these LP well-posedness have also been established through appropriate solution maps. Here, it is important to note that these LP well-posedness are independent of the non emptiness of the interior of the ordering cone and hence can be used in both solid and nonsolid optimization. Further, the Painlevé–Kuratowski convergence of the sets of optimal solutions of scalar problem which is defined by using generalized oriented distance function [27] to the set of minimal solutions of the set optimization problem is given. Finally, the stability of the set optimization problem is investigated firstly, by taking into account the perturbations of the ordering cone and then perturbations of both the ordering cone and the feasible set where the ordering cone is assumed to have nonempty interior. The Painlevé-Kuratowski convergence of a sequence of minimal solution sets of perturbed problems to the minimal solution sets of the original problem have been studied. Some of these results generalize the corresponding results of [3] from the vector optimization setting to the set optimization case.

2 Preliminaries

Assume that X and Y are real normed spaces and L is a convex closed cone in Y. Let $B(0, \epsilon)$ denote an open ball with center origin and radius $\epsilon > 0$ in the relevant spaces X and Y, M^c the complement of a set M and $\mathcal{P}(Y) = \{M \subseteq Y : M \neq \emptyset\}$.

Throughout, we consider the following upper set-relation on $\mathcal{P}(Y)$ [17]. For any $M, N \in \mathcal{P}(Y)$

$$M \preccurlyeq^u_L N \Leftrightarrow M \subseteq N - L.$$

We say that

$$M \sim^u N \Leftrightarrow M \preccurlyeq^u_L N \text{ and } N \preccurlyeq^u_L M.$$

Consider a set-valued map $F : X \rightrightarrows Y$. We denote by dom F the set $\{x \in X : F(x) \neq \emptyset\}$.

Now, we recall few basic definitions.

Definition 2.1 [6] A set $M \in \mathcal{P}(Y)$ is called $(-L)$-closed and $(-L)$-proper if $M - L$ is closed and $M - L \neq Y$, respectively.

Definition 2.2 [6] For a nonempty set $A \subseteq X$, the mapping F is called

(i) $(-L)$-closed-valued on A if for each $x \in A$, $F(x)$ is $(-L)$-closed.
(ii) $(-L)$-proper-valued on A if for each $x \in A$, $F(x)$ is $(-L)$-proper.

Definition 2.3 [15] The mapping F is called

(i) upper semicontinuous (u.s.c.) at $\hat{x} \in X$ if for every open set V in Y with $F(\hat{x}) \subseteq V$, there exists a neighbourhood W of $\hat{x}$ such that for every $x \in W$, $F(x) \subseteq V$.
(ii) lower semicontinuous (l.s.c.) at $\hat{x} \in X$ if for every open set V in Y with $F(\hat{x}) \cap V \neq \emptyset$, there exists a neighbourhood W of $\hat{x}$ such that for every $x \in W$, $F(x) \cap V \neq \emptyset$.
(iii) Hausdorff upper semicontinuous (H-u.s.c.) at $\hat{x} \in X$ if for any neighbourhood U of 0_Y, a neighbourhood W of $\hat{x}$ can be found such that $F(x) \subseteq F(\hat{x}) + U$, for every $x \in W$.
(iv) closed at $\hat{x} \in X$ if for any sequence $((x_n, y_n))$ in graph $F = \{(x, y) \in X \times Y : y \in F(x)\}$ which converges to $(\hat{x}, \hat{y})$ implies that $\hat{y} \in F(\hat{x})$.

The map F satisfies a property on a nonempty set $A \subseteq X$ if it satisfies that property for every point in A.

The following sequential criterion of upper and lower semicontinuity will be used.

Theorem 2.1 *(i) [15] If $\hat{x} \in X$ and $F(\hat{x})$ is compact, then the mapping F is u.s.c. at $\hat{x}$ if and only if for any sequence $(x_n) \in X$ which converges to $\hat{x}$ and for any $y_n \in F(x_n)$, there exists a subsequence (y_{n_k}) of (y_n) with $y_{n_k} \to \hat{y} \in F(\hat{x})$.*
(ii) [1] The mapping F is l.s.c. at $\hat{x} \in X$ if and only if for any sequence $(x_n) \in X$ which converges to $\hat{x}$ and for any $\hat{y} \in F(\hat{x})$, there exists a sequence $(y_n) \in F(x_n)$ which converges to $\hat{y}$.

Now, we recall the following definition of covergence of a sequence of sets.

Definition 2.4 [15] Let (M_n) be a sequence of sets in X. Let

$$\text{Li } M_n = \{x \in X : x_n \to x, x_n \in M_n, \text{for sufficiently large } n\},$$
$$\text{Ls } M_n = \{x \in X : x_{n_k} \to x, x_{n_k} \in M_{n_k}, n_k \text{ is an increasing sequence of integers}\}.$$

We say that (M_n) converges to $M \subseteq X$ in the sense of Painlevé-Kuratowski convergence if Ls $M_n \subseteq M \subseteq$ Li M_n. The relation Ls $M_n \subseteq M$ is referred to as the upper part of the convergence and the relation $M \subseteq$ Li M_n is referred to as the lower part of the convergence.

Definition 2.5 [12] The oriented distance function $\Delta_M : Y \to \mathbb{R} \cup \{\pm\infty\}$ is defined as

$$\Delta_M(y) = d(y, M) - d(y, M^c), \quad y \in Y,$$

where $M \subseteq Y$ and $d(y, M) = \inf_{m\in M} \|y - m\|$. If $M = \emptyset$ then $d(y, \emptyset) = +\infty$.

For $M \subseteq Y$, we consider a generalization $\mathcal{D}_M : \mathcal{P}(Y) \to \mathbb{R}\cup\{\pm\infty\}$ of the oriented distance function Δ_M, introduced in [27], defined as follows:

$$\mathcal{D}_M(N) = \sup_{n\in N} \Delta_M(n), \quad N \in \mathcal{P}(Y).$$

We refer the following lemma from [6].

Lemma 2.1 *Let $N \in \mathcal{P}(Y)$, M be a closed set in $\mathcal{P}(Y)$ and $r \in \mathbb{R}$. Then the following hold:*

(i) $N \subseteq r\bar{B} + M$ if $\mathcal{D}_M(N) \le r$;
(ii) $\mathcal{D}_M(N) \le r$ if $r \ge 0$ and $N \subseteq r\bar{B} + M$;

where $\bar{B}$ is a closed unit ball in Y.

Now, the following constraint set-valued optimization problem $(P_{L,A})$ is considered.

$$(P_{L,A}) \qquad \begin{array}{c} \min_L \ F(x) \\ \text{subject to} \quad x \in A \end{array}$$

where A is a nonempty set in X.

Definition 2.6 [17] An element $\hat{x} \in A$ is called a u-minimal solution of $(P_{L,A})$ if $x \in A$, $F(x) \preccurlyeq^u_L F(\hat{x}) \Rightarrow F(\hat{x}) \preccurlyeq^u_L F(x)$. Let the set of all u-minimal solutions be denoted by $\mathrm{Min}_L(A, F)$.

Throughout the paper, it is assumed that $\mathrm{Min}_L(A, F) \neq \emptyset$. We remark that, if $\hat{x} \in \mathrm{Min}_L(A, F)$ and x is an element of A such that $F(x) \preccurlyeq^u_L F(\hat{x})$, then $x \in \mathrm{Min}_L(A, F)$.
Consider the following scalar optimization problem:

$$(P) \qquad \begin{array}{c} \min \quad \mathcal{D}_{F(v)-L}(F(x)) \\ \text{subject to} \quad x \in A \end{array}$$

where $v \in \mathrm{Min}_L(A, F)$. Let $\mathrm{argmin}(A, \mathcal{D}_{F(v)-L} \circ F)$ be the set of all minimal solutions of (P).
A link between minimal solutions of $(P_{L,A})$ and (P) is recalled from [6].

Theorem 2.2 *[6] Let $v \in \mathrm{Min}_L(A, F)$, $F(v)$ be $(-L)$-proper and $(-L)$-closed set. Then $\mathrm{argmin}(A, \mathcal{D}_{F(v)-L} \circ F) \subseteq \mathrm{Min}_L(A, F)$ and the converse inclusion hold provided $F(y) \sim^u F(v)$, for every $y \in \mathrm{Min}_L(A, F)$.*

Lemma 2.2 [6] *Let* $v \in \mathrm{Min}_L(A, F)$, $F(v)$ *be* $(-L)$*-proper and* $(-L)$*-closed set. Then* $\mathcal{D}_{F(v)-L}(F(x)) \geq 0$, *for every* $x \in A$.

We next recall u-domination property from [11].

Definition 2.7 The problem $(P_{L,A})$ satisfies the u-domination property if for every $x \in A$, there exists $\hat{x} \in \mathrm{Min}_L(A, F)$ such that $F(\hat{x}) \preccurlyeq_L^u F(x)$.

3 *LP* well-posedness of $(P_{L,A})$

In this section, LP well-posedness is introduced for $(P_{L,A})$ in both the set sense and scalar sense. A Relationship between them is established. Further, necessary and sufficient conditions are established for the LP well-posedness in the set sense. A comparison is drawn between the LP well-posedness in the set sense and the generalized LP well-posedness defined in [16].

Definition 3.1 A sequence $(x_n) \in X$ is called a LP minimizing sequence for the problem $(P_{L,A})$ in the set sense if there exists a real sequence (ϵ_n), $\epsilon_n \geq 0$, $\epsilon_n \to 0$ as $n \to +\infty$ and $v_n \in \mathrm{Min}_L(A, F)$ such that $x_n \in A + B(0, \epsilon_n)$ and $F(x_n) \preccurlyeq_L^u F(v_n) + \epsilon_n \bar{B}$ for all n.

Definition 3.2 The problem $(P_{L,A})$ is called LP well-posed in the set sense if every LP minimizing sequence (x_n) for the problem $(P_{L,A})$ in the set sense has a subsequence (x_{n_k}) such that $d(x_{n_k}, \mathrm{Min}_L(A, F)) \to 0$.

Remark 3.1 The above Definition 3.2 of LP well-posedness defined for $(P_{L,A})$ extends the idea of generalized well-posedness defined in [6] by allowing minimizing sequences to lie outside the constraint set A.

Clearly, $F(x_n) \preccurlyeq_L^u F(v_n) + \epsilon_n \bar{B} \Leftrightarrow \mathcal{D}_{F(v_n)-L}(F(x_n)) \leq \epsilon_n$ provided F is $(-L)$-closed-valued on $\mathrm{Min}_L(A, F)$.

Motivated by this fact and Definitions 3.3 and 3.4 of [3], we define the following notion of LP well-posedness in the scalar sense.

Definition 3.3 A sequence $(x_n) \in X$ is called a LP minimizing sequence for the problem $(P_{L,A})$ with respect to $(v_n) \in \mathrm{Min}_L(A, F)$ in the scalar sense if there exists a real sequence (ϵ_n), $\epsilon_n \geq 0$, $\epsilon_n \to 0$ such that $x_n \in A + B(0, \epsilon_n)$ and $\mathcal{D}_{F(v_n)-L}(F(x_n)) \leq \epsilon_n$.

Definition 3.4 The problem $(P_{L,A})$ is called LP well-posed in the scalar sense if for every LP minimizing sequence (x_n) for $(P_{L,A})$ with respect to $(v_n) \in \mathrm{Min}_L(A, F)$ in the scalar sense, there exist subsequences (x_{n_k}) of (x_n) and (v_{n_k}) of (v_n) such that $v_{n_k} \to \hat{v}$ and $x_{n_k} \to \hat{x}$ for some $\hat{x} \in \mathrm{argmin}(A, \mathcal{D}_{F(\hat{v})-L} \circ F)$.

The following example illustrates these definitions.

Example 3.1 (i) Let $X = \mathbb{R}$, $A = [0, 1] \subseteq \mathbb{R}$, $Y = \mathbb{R}^2$ and $L = \mathbb{R}^2_+ \subseteq \mathbb{R}^2$. Consider $F : X \rightrightarrows Y$ defined as

$$F(x) = \begin{cases} (|x|, 1) - \mathbb{R}^2_+, & \text{if } x < 0, \\ (0, 1) - \mathbb{R}^2_+, & \text{if } x = 0, \\ (1, 1) - \mathbb{R}^2_+, & \text{if } 0 < x < 1, \\ (0, 1) - \mathbb{R}^2_+, & \text{if } x = 1, \\ (x - 1, 1) - \mathbb{R}^2_+, & \text{if } x > 1. \end{cases}$$

Here, $\mathrm{Min}_L(A, F) = \{0, 1\}$. Clearly, $(P_{L,A})$ is LP well-posed in the scalar sense.

(ii) Let $X = \mathbb{R}$, $A = [0, 2] \subseteq \mathbb{R}$, $Y = l^2$ and $L = l^2_+ \subseteq l^2$. Consider $F : X \rightrightarrows Y$ defined as

$$F(x) = \begin{cases} (2, 0, 0, \ldots) - l^2_+, & \text{if } x < 0, \\ (0, 5, 0, 0, \ldots) - l^2_+, & \text{if } 0 \leq x \leq 1, \\ \{(x_n) \in l^2 : 0 < x_1 \leq 1\}, & \text{if } 1 < x < 2, \\ (0, 5, 0, 0, \ldots) - l^2_+, & \text{if } x = 2, \\ \{(x - 2, 4, 0, 0, \ldots)\}, & \text{if } x > 2. \end{cases}$$

Here, $\mathrm{Min}_L(A, F) = [0, 1] \cup \{2\}$. It is observed that $(P_{L,A})$ is LP well-posed in the set sense.

(iii) Let $X = \mathbb{R}$, $A = [0, 2] \subseteq \mathbb{R}$, $Y = l^2$ and $L = l^2_+ \subseteq l^2$. Consider $F : X \rightrightarrows Y$ defined as

$$F(x) = \begin{cases} (2, 0, 0, \ldots) - l^2_+, & \text{if } x < 0, \\ (0, 5, 0, 0, \ldots) - l^2_+, & \text{if } 0 \leq x \leq 1, \\ \{(x_n) \in l^2 : 0 < x_1 \leq 1\}, & \text{if } 1 < x \leq 2, \\ \{(x - 2, 4, 0, 0, \ldots)\}, & \text{if } x > 2. \end{cases}$$

Then, $\mathrm{Min}_L(A, F) = [0, 1]$. We observe that $(P_{L,A})$ is neither LP well-posed in the set sense nor LP well-posed in the scalar sense.

We now prove a relationship between LP well-posedness in the set sense and scalar sense.

Theorem 3.1 *Suppose that* $\mathrm{Min}_L(A, F)$ *is compact and* F *is* $(-L)$*-proper-valued and* $(-L)$*-closed-valued on* $\mathrm{Min}_L(A, F)$*. If* $(P_{L,A})$ *is* LP *well-posed in the scalar sense then it is* LP *well-posed in the set sense. The converse holds provided* $F(u) \sim^u F(v)$*, for all* $u, v \in \mathrm{Min}_L(A, F)$.

Proof Suppose that $(P_{L,A})$ is LP well-posed in the scalar sense and $(x_n) \in X$ is a LP minimizing sequence for $(P_{L,A})$ in the set sense. Then, there exist $\epsilon_n \geq 0$, $\epsilon_n \to 0$ and $v_n \in \mathrm{Min}_L(A, F)$ such that $x_n \in A + B(0, \epsilon_n)$ and $\mathcal{D}_{F(v_n)-L}(F(x_n)) \leq \epsilon_n$. Thus, (x_n) is a LP minimizing sequence for $(P_{L,A})$ with respect to (v_n) in the scalar sense and therefore there exist subsequences (x_{n_k}) of (x_n) and (v_{n_k}) of (v_n) such that $v_{n_k} \to \hat{v}$ and $x_{n_k} \to \hat{x}$ for some $\hat{x} \in \operatorname{argmin}(A, \mathcal{D}_{F(\hat{v})-L} \circ F)$. It follows from Theorem 2.2 that $\hat{x} \in \operatorname{Min}_L(A, F)$ which further implies that $d(x_{n_k}, \operatorname{Min}_L(A, F)) \to 0$. Hence, $(P_{L,A})$ is LP well-posed in the set sense.

Conversely, suppose that $(x_n) \in X$ is a LP minimizing sequence for $(P_{L,A})$ with respect to $(v_n) \in \operatorname{Min}_L(A, F)$ in the scalar sense. Then, there exist $\epsilon_n \geq 0$, $\epsilon_n \to 0$ such that $x_n \in A + B(0, \epsilon_n)$ and $\mathcal{D}_{F(v_n)-L}(F(x_n)) \leq \epsilon_n$. This implies that (x_n) is a LP minimizing sequence for $(P_{L,A})$ in the set sense. As $\operatorname{Min}_L(A, F)$ is compact, there exists a subsequence (v_{n_k}) of (v_n) such that $v_{n_k} \to \hat{v}$ for some $\hat{v} \in \operatorname{Min}_L(A, F)$. Since $(P_{L,A})$ is LP well-posed in the set sense, therefore there exists a subsequence $(x_{n_{k_l}})$ of (x_{n_k}) such that $d(x_{n_{k_l}}, \operatorname{Min}_L(A, F)) \to 0$. From the compactness of $\operatorname{Min}_L(A, F)$ and Theorem 2.2, it follows that there exist a subsequence $(x'_{n_{k_l}})$ of $(x_{n_{k_l}})$ and $\hat{x} \in \operatorname{argmin}(A, \mathcal{D}_{F(\hat{v})-L} \circ F)$ such that $x'_{n_{k_l}} \to \hat{x}$. Hence, $(P_{L,A})$ is LP well-posed in the scalar sense. □

Remark 3.2 (i) Theorem 3.1 is a generalized version of Theorem 3.2 of [3] in set-valued setting.

(ii) In example 3.1 (i), $\operatorname{Min}_L(A, F)$ is compact and F is $(-L)$-proper-valued and $(-L)$-closed-valued on $\operatorname{Min}_L(A, F)$. Also, $(P_{L,A})$ is LP well-posed in the scalar sense. Therefore, from Theorem 3.1 it follows that $(P_{L,A})$ is LP well-posed in the set sense.

(iii) In example 3.1 (ii), $\operatorname{Min}_L(A, F)$ is compact and F is $(-L)$-proper-valued and $(-L)$-closed-valued on $\operatorname{Min}_L(A, F)$. Also, $F(u) \sim^u F(v)$, for all $u, v \in \operatorname{Min}_L(A, F)$ and $(P_{L,A})$ is LP well-posed in the set sense. Therefore, applying Theorem 3.1, $(P_{L,A})$ is LP well-posed in the scalar sense.

The following example justifies that the converse of Theorem 3.1 may not hold in the absence of the condition $F(u) \sim^u F(v)$, for all $u, v \in \operatorname{Min}_L(A, F)$.

Example 3.2 Let $X = \mathbb{R}$, $A = [0, 1] \subseteq \mathbb{R}$, $Y = \mathbb{R}^2$, $L = \mathbb{R}^2_+ \subseteq \mathbb{R}^2$. Consider $F : X \rightrightarrows Y$ defined as

$$F(x) = \begin{cases} \{(1, 1)\}, & \text{if } x < 0, \\ (0, 1) - \mathbb{R}^2_+, & \text{if } x = 0, \\ (1, 1) - \mathbb{R}^2_+, & \text{if } 0 < x < 1, \\ (1, 0) - \mathbb{R}^2_+, & \text{if } x = 1, \\ (x - 1) - \mathbb{R}^2_+, & \text{if } x > 1. \end{cases}$$

Here, $\operatorname{Min}_L(A, F) = \{0, 1\}$. Clearly, $F(0) \nsim^u F(1)$ and all other conditions of Theorem 3.1 are satisfied. We observe that the problem $(P_{L,A})$ is LP well-posed in the set sense but not LP well-posed in the scalar sense.

We now obtain sufficiency and necessary criteria for the LP well-posedness in the set sense.

Theorem 3.2 *Suppose that A is a compact set. If F is l.s.c on $A \setminus \mathrm{Min}_L(A, F)$ and $(-L)$-closed-valued on $\mathrm{Min}_L(A, F)$, $(P_{L,A})$ satisfies u-domination property and*

$$F(u) \sim^u F(v), \text{ for any } u, v \in \mathrm{Min}_L(A, F), \tag{1}$$

then the problem $(P_{L,A})$ is LP well-posed in the set sense.

Proof Suppose that (x_n) is a LP minimizing sequence for $(P_{L,A})$ in the set sense. Then, there exist $\epsilon_n \geq 0$, $\epsilon_n \to 0$ and $v_n \in \mathrm{Min}_L(A, F)$ such that $x_n \in A + B(0, \epsilon_n)$ and

$$F(x_n) \subseteq F(v_n) + \epsilon_n \bar{B} - L. \tag{2}$$

Since A is compact, therefore there exist a subsequence (x_{n_k}) of (x_n) and $\hat{x} \in A$ such that $x_{n_k} \to \hat{x}$. If $\hat{x} \in \mathrm{Min}_L(A, F)$ then $d(x_{n_k}, \mathrm{Min}_L(A, F)) \to 0$.

If $\hat{x} \notin \mathrm{Min}_L(A, F)$ then there exists a point $x \in A$ such that $F(x) \preccurlyeq^u_L F(\hat{x})$ but $F(\hat{x}) \not\preccurlyeq^u_L F(x)$. Using u-domination property of $(P_{L,A})$, there exists a point $v \in \mathrm{Min}_L(A, F)$ such that $F(v) \preccurlyeq^u_L F(x)$ which further implies that $F(v) \preccurlyeq^u_L F(\hat{x})$. Also, $F(\hat{x}) \not\preccurlyeq^u_L F(v)$ because if $F(\hat{x}) \preccurlyeq^u_L F(v)$ then $F(\hat{x}) \preccurlyeq^u_L F(x)$, which is a contradiction. Therefore, without loss of generality, we can assume that $x \in \mathrm{Min}_L(A, F)$. Since $F(\hat{x}) \not\preccurlyeq^u_L F(x)$, therefore there exists a point $\hat{y} \in F(\hat{x})$ such that

$$\hat{y} \notin F(x) - L. \tag{3}$$

From the lower semicontinuity of F at $\hat{x}$, it follows that there exist $y_{n_k} \in F(x_{n_k})$ such that $y_{n_k} \to \hat{y}$. By (2) together with the condition (1), it can be shown that $\hat{y} \in F(x) - L$ which contradicts (3). Hence, $(P_{L,A})$ is LP well-posed in the set sense. □

Remark 3.3 In example 3.1 (iii), F is not l.s.c. at $x = 2$. The other conditions of Theorem 3.2 are satisfied and $(P_{L,A})$ is not LP well-posed in the set sense. Therefore, lower semicontinuity of F cannot be dropped in Theorem 3.2.

The following examples justify that Theorem 3.2 may not hold without compactness of A, u-domination property of $(P_{L,A})$ and condition (1).

Example 3.3 Let $X = \mathbb{R}$, $A = [0, 2) \subseteq \mathbb{R}$, $Y = l^2$ and $L = l^2_+ \subseteq l^2$. Consider $F : X \rightrightarrows Y$ defined as

$$F(x) = \begin{cases} (2, 0, 0, \ldots) - l^2_+, & \text{if } x < 0, \\ (0, 5, 0, 0, \ldots) - l^2_+, & \text{if } 0 \leq x \leq 1, \\ \{(x_n) \in l^2 : 0 < x_1 \leq 1\}, & \text{if } 1 < x < 2, \\ \{(2, 4, 0, 0, \ldots)\}, & \text{if } x = 2, \\ \{(x - 2, 4, 0, 0, \ldots)\}, & \text{if } x > 2. \end{cases}$$

Here, $\mathrm{Min}_L(A, F) = [0, 1]$. Clearly, A is not compact, while all other assumptions in Theorem 3.2 hold. It is observed that $(P_{L,A})$ is not LP well-posed in the set sense. Indeed, if $x_n = 2 + \frac{1}{2n}$, $v_n = \frac{1}{n}$ and $\epsilon_n = \frac{1}{n}$ for all $n \in \mathbb{N}$ then (x_n) is a LP minimizing sequence for $(P_{L,A})$ in the set sense but $d(x_n, \mathrm{Min}_L(A, F)) \to 1$.

Example 3.4 Let $X = \mathbb{R}$, $A = [0, 2] \subseteq \mathbb{R}$, $Y = \mathbb{R}^2$ and $L = \mathbb{R}^2_+ \subseteq \mathbb{R}^2$. Consider $F : X \rightrightarrows Y$ defined as

$$F(x) = \begin{cases} [0, 1] \times [0, 1], & \text{if } 0 \leq x \leq 1, \\ (x, x-1) - \mathbb{R}^2_+, & \text{if } 1 < x < 2, \\ [1, 2] \times [0, 1], & \text{if } x = 2, \\ [1, x-1] \times [0, 1], & \text{if } 2 < x < 3, \\ \{(2, 2)\}, & \text{otherwise.} \end{cases}$$

We observe that $\text{Min}_L(A, F) = [0, 1]$ and all assumptions of Theorem 3.2 hold but u-domination property is not satisfied. It can be seen that $(P_{L,A})$ is not LP well-posed in the set sense. Indeed, if $x_n = 2 + \frac{1}{2n}$, $v_n = \frac{1}{n}$ and $\epsilon_n = \frac{1}{n}$ for all $n \in \mathbb{N}$ then (x_n) is a LP minimizing sequence for $(P_{L,A})$ in the set sense but $d(x_n, \text{Min}_L(A, F)) \to 1$.

Example 3.5 Let $X = \mathbb{R}$, $A = [0, 2] \subseteq \mathbb{R}$, $Y = \mathbb{R}^2$, dom $F = [0, 3]$ and $L = \mathbb{R}^2_+ \subseteq \mathbb{R}^2$. Consider $F : X \rightrightarrows Y$ defined as

$$F(x) = \begin{cases} [0, 1] \times [0, 1], & \text{if } x = 0, \\ [0, 1+x] \times [0, 1-x], & \text{if } 0 < x < 1, \\ [0, 2] \times \{0\}, & \text{if } x = 1, \\ [1, 2] \times [0, 1], & \text{if } 1 < x \leq 2, \\ [1, x-1] \times [0, 1], & \text{if } 2 < x \leq 3. \end{cases}$$

It is observed that $\text{Min}_L(A, F) = [0, 1]$. Here, all assumptions of Theorem 3.2 are satisfied but condition (1) for $u = 0$ and $v = 1$ and the conclusion of Theorem 3.2 fails to hold. Indeed, if $x_n = 2 + \frac{1}{2n}$, $v_n = 0$ and $\epsilon_n = \frac{1}{n}$ for all $n \in \mathbb{N}$ then (x_n) is a LP minimizing sequence for $(P_{L,A})$ in the set sense but $d(x_n, \text{Min}_L(A, F)) \to 1$.

Theorem 3.3 *Suppose that* $\text{Min}_L(A, F)$ *is compact,* F *is closed on* $\text{Min}_L(A, F)$ *and the problem* $(P_{L,A})$ *is* LP *well-posed in the set sense. Then, for any* LP *minimizing sequence* (x_n) *for* $(P_{L,A})$ *in the set sense there exists a subsequence* (x_{n_k}) *of* (x_n) *such that* $\text{Ls } F(x_{n_k}) \subseteq F(\text{Min}_L(A, F))$.

Proof Suppose that (x_n) is any LP minimizing sequence for $(P_{L,A})$ in the set sense. As the problem $(P_{L,A})$ is LP well-posed in the set sense, there exists a subsequence (x_{n_k}) of (x_n) such that $d(x_{n_k}, \text{Min}_L(A, F)) \to 0$. From the compactness of the set $\text{Min}_L(A, F)$, it follows that (x_{n_k}) has a subsequence which is again denoted by (x_{n_k}) converging to $\hat{x} \in \text{Min}_L(A, F)$. If $\hat{y} \in \text{Ls } F(x_{n_k})$ then there exist $y_{n_{k_l}} \in F(x_{n_{k_l}})$ such that $y_{n_{k_l}} \to \hat{y}$. Using the closedness of F at $\hat{x}$, it follows that $\hat{y} \in F(\hat{x}) \subseteq F(\text{Min}_L(A, F))$. Hence, $\text{Ls } F(x_{n_k}) \subseteq F(\text{Min}_L(A, F))$. □

We observe that the generalized LP well-posedness introduced in [16] assumes that L has nonempty interior. In order to compare LP well-posedness in the set sense defined in this paper with the well-posedness defined in [16], we also assume that L has nonempty interior in the rest of this section.

Khoshkhabar-amiranloo [16] defined a notion of generalized LP well-posedness for $(P_{L,A})$ using the lower set-relation with the assumption that int$L$ is nonempty. We recall the corresponding notion of the well-posedness for $(P_{L,A})$ using the upper set-relation.

Definition 3.5 [16] Let $e \in \text{int}L$. A sequence (x_n) in X is said to be a generalized LP minimizing sequence for $(P_{L,A})$ if there exist a real sequence $(\epsilon_n), \epsilon_n \geq 0,\ \epsilon_n \to 0$ and a sequence (v_n) in $\text{Min}_L(A, F)$ such that $d(x_n, A) \leq \epsilon_n$ and $F(x_n) \preccurlyeq^u_L F(v_n) + \epsilon_n e$. The problem $(P_{L,A})$ is said to be generalized LP well-posed if every generalized LP minimizing sequence (x_n) has a subsequence (x_{n_k}) such that $x_{n_k} \to \hat{x}$, for some $\hat{x} \in \text{Min}_L(A, F)$.

In [16] it has been mentioned that the above Definition 3.5 is independent of the choice of the vector $e \in \text{int}L$. But it is clear that this definition is still dependent on the cone L. Here, we show that the Definition 3.5 does not depend on L at all.

Proposition 3.1 *Let $e \in \text{int}L$. A sequence (x_n) in X is a LP minimizing sequence for $(P_{L,A})$ in the set sense if and only if it is generalized LP minimizing sequence for $(P_{L,A})$.*

Proof Suppose that (x_n) is a LP minimizing sequence for $(P_{L,A})$ in the set sense. Then there exist $\epsilon_n \geq 0,\ \epsilon_n \to 0$ and a sequence $(v_n) \in \text{Min}_L(A, F)$ such that $x_n \in A + B(0, \epsilon_n)$ and $F(x_n) \subseteq F(v_n) + \epsilon_n \bar{B} - L$. Clearly, there always exists a sequence $(\alpha_n), \alpha_n \geq 0,\ \alpha_n \to 0$ such that $\epsilon_n \bar{B} \subseteq \alpha_n e - L$ which in turn implies that (x_n) is a generalized LP minimizing sequence for $(P_{L,A})$.

Conversely, if (x_n) is a generalized LP minimizing sequence for $(P_{L,A})$, then there exist $\epsilon_n \geq 0,\ \epsilon_n \to 0$ and a sequence $(v_n) \in \text{Min}_L(A, F)$ such that $d(x_n, A) \leq \epsilon_n$ and $F(x_n) \preccurlyeq^u_L F(v_n) + \epsilon_n e$. Let $\alpha_n = \|\epsilon_n e\|$. Then $\alpha_n \geq 0,\ \alpha_n \to 0$ and $\epsilon_n e \in \alpha_n \bar{B}$. This implies that (x_n) is a LP minimizing sequence for $(P_{L,A})$ in the set sense. □

Theorem 3.4 *Suppose that $e \in \text{int}L$. If $(P_{L,A})$ is generalized LP well-posed then it is LP well-posed in the set sense. The converse hold only if* $\text{Min}_L(A, F)$ *is compact.*

Remark 3.4 The converse of Theorem 3.4 may not be true if $\text{Min}_L(A, F)$ is not compact which can be verified by the following example.

Example 3.6 Let $X = \mathbb{R}$, $A = [-1, 1] \subseteq \mathbb{R}$, $Y = \mathbb{R}^2$, $L = \mathbb{R}^2_+ \subseteq \mathbb{R}^2$ and $e = (1, 1)$. Consider $F : X \rightrightarrows Y$ defined as

$$F(x) = \begin{cases} (0,1) - \mathbb{R}^2_+, & \text{if } 0 < x \leq 1, \\ (1,1) - \mathbb{R}^2_+, & \text{otherwise.} \end{cases}$$

Here, $\text{Min}_L(A, F) = (0, 1]$. Clearly, $\text{Min}_L(A, F)$ is not compact and the problem $(P_{L,A})$ is LP well-posed in the set sense but not generalized LP well-posed. Indeed, if $x_n = 1/n$, $v_n = 1/n$ and $\epsilon_n = 1/n$ for all $n \in \mathbb{N}$ then (x_n) is a generalized LP minimizing sequence for $(P_{L,A})$ but every convergent subsequence of (x_n) converges to $0 \notin \text{Min}_L(A, F)$.

4 *LP* well-posedness and approximate solution map

In this section, some characterizations of LP well-posedness for $(P_{L,A})$ in terms of closedness and Hausdorff upper semicontinuity of the approximate solution set-valued maps are given.

For this, we consider the following set-valued maps.

Define $G : \mathbb{R}_+ \times \mathrm{Min}_L(A, F) \rightrightarrows X$ by

$$G(\epsilon, v) = \{x \in X : x \in A + B(0, \epsilon),\ \mathcal{D}_{F(v)-L}(F(x)) \leq \epsilon\},$$

where $\epsilon \geq 0$ and $v \in \mathrm{Min}_L(A, F)$.

It follows from Lemma 2.2 that $G(0, v) = \mathrm{argmin}(A, \mathcal{D}_{F(v)-L} \circ F)$ provided F is $(-L)$-proper-valued and $(-L)$-closed-valued on $\mathrm{Min}_L(A, F)$.

Define $D : \mathbb{R}_+ \rightrightarrows X$ by

$$D(\epsilon) = \bigcup_{y \in \mathrm{Min}_L(A,F)} \{x \in X : x \in A + B(0, \epsilon), F(x) \preccurlyeq^u_L F(y) + \epsilon \bar{B}\},\ \epsilon \in \mathbb{R}_+.$$

Here, D and $D(\epsilon)$ denote the approximate solution map and the set of approximate solutions, respectively. We observe that $D(0) \subseteq D(\epsilon)$ for all $\epsilon > 0$ and $D(0) = \mathrm{Min}_L(A, F)$.

Theorem 4.1 *If the problem $(P_{L,A})$ is LP well-posed in the scalar sense then G is closed at $(0, v)$ for every $v \in \mathrm{Min}_L(A, F)$. The converse holds if F is $(-L)$-proper-valued and $(-L)$-closed-valued on $\mathrm{Min}_L(A, F)$, and $\mathrm{Min}_L(A, F)$ and A are compact sets.*

Proof Let $v \in \mathrm{Min}_L(A, F)$ be arbitrary. Let $(\epsilon_n, v_n) \in \mathbb{R}_+ \times \mathrm{Min}_L(A, F)$ such that $(\epsilon_n, v_n) \to (0, v)$ and $x_n \in G(\epsilon_n, v_n)$ with $x_n \to x$. It follows that (x_n) is a LP minimizing sequence for $(P_{L,A})$ with respect to (v_n) in the scalar sense. As $(P_{L,A})$ is LP well-posed in the scalar sense, there exist subsequences (v_{n_k}) of (v_n) and (x_{n_k}) of (x_n) such that $v_{n_k} \to \hat{v}$ and $x_{n_k} \to \hat{x}$ where $\hat{x} \in \mathrm{argmin}(A, \mathcal{D}_{F(\hat{v})-L} \circ F)$ which further implies that $\hat{v} = v$ and $\hat{x} = x$. Thus, $x \in G(0, v)$ and so G is closed at $(0, v)$.

Conversely, if (x_n) is a LP minimizing sequence for $(P_{L,A})$ with respect to $(v_n) \in \mathrm{Min}_L(A, F)$ in the scalar sense, then there exist $\epsilon_n \geq 0$, $\epsilon_n \to 0$ such that $x_n \in A + B(0, \epsilon_n)$ and $\mathcal{D}_{F(v_n)-L}(F(x_n)) \leq \epsilon_n$. This implies that $x_n \in G(\epsilon_n, v_n)$. On using the compactness of the sets $\mathrm{Min}_L(A, F)$ and A and closedness of G, we obtain that there exist subsequences (v_{n_k}) and (x_{n_k}) such that $v_{n_k} \to \hat{v}$ and $x_{n_k} \to \hat{x}$ where $\hat{v} \in \mathrm{Min}_L(A, F)$ and $\hat{x} \in G(0, \hat{v}) = \mathrm{argmin}(A, \mathcal{D}_{F(\hat{v})-L} \circ F)$. Hence, $(P_{L,A})$ is LP well-posed in the scalar sense. □

The proof of the following corollary follows by invoking Theorems 3.1 and 4.1.

Corollary 4.1 *Suppose that $\mathrm{Min}_L(A, F)$ is compact, F is $(-L)$-proper-valued and $(-L)$-closed-valued on $\mathrm{Min}_L(A, F)$. Then the following hold:*

(i) If A is compact and G is closed at $(0, v)$ for every $v \in \mathrm{Min}_L(A, F)$ then $(P_{L,A})$ is LP well-posed in the set sense.

(ii) If $(P_{L,A})$ *is* LP *well-posed in the set sense and* $F(u) \sim^u F(v)$*, for any* $u, v \in \mathrm{Min}_L(A, F)$ *then* G *is closed at* $(0, v)$ *for every* $v \in \mathrm{Min}_L(A, F)$.

Remark 4.1 The above two results of this section generalize Theorem 4.1 of [3] from the vector-valued setting to set-valued setting.

Theorem 4.2 *If* D *is* H-u.s.c. *at* 0 *then* $(P_{L,A})$ *is* LP *well-posed in the set sense. The converse holds if* $\mathrm{Min}_L(A, F)$ *is compact.*

Proof If (x_n) is a LP minimizing sequence for $(P_{L,A})$ in the set sense, then there exist $\epsilon_n \geq 0$, $\epsilon_n \to 0$ and $v_n \in \mathrm{Min}_L(A, F)$ such that $x_n \in A + B(0, \epsilon_n)$ and $F(x_n) \preccurlyeq_L^u F(v_n) + \epsilon_n \bar{B}$. Let $\epsilon > 0$ be arbitrary. Since D is H-u.s.c. at 0, therefore there exists a neighbourhood W of 0 such that $D(\alpha) \subseteq D(0) + B(0, \epsilon)$ for all $\alpha \in W$ and hence there exists $m \in \mathbb{N}$ such that $D(\epsilon_n) \subseteq D(0) + B(0, \epsilon)$ for all $n \geq m$. This implies that $x_n \in D(0) + B(0, \epsilon)$ for all $n \geq m$ which in turn implies that $d(x_n, \mathrm{Min}_L(A, F)) \to 0$. Hence, $(P_{L,A})$ is LP well-posed in the set sense.

Conversely, if D is not H-u.s.c. at 0 then there exists an $\epsilon > 0$ such that for every neighbourhood W of 0 there exists an $\alpha > 0$ with $D(\alpha) \nsubseteq D(0) + B(0, \epsilon)$. So, we can find $\alpha_n > 0$, $\alpha_n \to 0$ such that $D(\alpha_n) \nsubseteq D(0) + B(0, \epsilon)$. This implies that there exist $x_n \in D(\alpha_n)$ such that

$$x_n \notin D(0) + B(0, \epsilon). \tag{4}$$

Thus, (x_n) is a LP minimizing sequence for $(P_{L,A})$ in the set sense. So, in this case there exists a subsequence (x_{n_k}) of (x_n) such that $d(x_{n_k}, \mathrm{Min}_L(A, F)) \to 0$. Also, $\mathrm{Min}_L(A, F)$ being compact implies that there exist a subsequence $(x_{n_{k_l}})$ of (x_{n_k}) and $\hat{x} \in \mathrm{Min}_L(A, F) = D(0)$ such that $x_{n_{k_l}} \to \hat{x}$. This implies that

$$\hat{x} \in D(0) + B(0, \epsilon). \tag{5}$$

It follows from (4) that $x_{n_{k_l}} \in (D(0) + B(0, \epsilon))^c$ which is closed and hence $\hat{x} \in (D(0) + B(0, \epsilon))^c$ which contradicts (5). Hence, D is H-u.s.c. at 0. □

Theorem 4.3 *If* $(P_{L,A})$ *is* LP *well-posed in the set sense and* $\mathrm{Min}_L(A, F)$ *is compact then* D *is closed at* 0*. Conversely, If* D *is closed at* 0 *and* A *is compact then* $(P_{L,A})$ *is* LP *well-posed in the set sense.*

Proof Suppose that $(P_{L,A})$ is LP well-posed in the set sense and $(\epsilon_n, x_n) \in$ graph D such that $(\epsilon_n, x_n) \to (0, \hat{x})$. It follows that (x_n) is a LP minimizing sequence for $(P_{L,A})$ in the set sense. Therefore, there exists a subsequence (x_{n_k}) of (x_n) such that $d(x_{n_k}, \mathrm{Min}_L(A, F)) \to 0$ which further implies that there exists a subsequence $(x_{n_{k_l}})$ of (x_{n_k}) such that $x_{n_{k_l}} \to \hat{y}$ for some $\hat{y} \in \mathrm{Min}_L(A, F) = D(0)$. This implies that $\hat{x} \in D(0)$. Hence, D is closed at 0.

The converse follows on the similar lines as in the Theorem 3.3 of [16]. □

The following example illustrates Theorems 4.2 and 4.3.

Example 4.1 Let $X = \mathbb{R}$, $A = [0,1] \subseteq \mathbb{R}$, $Y = \mathbb{R}$, $L = \mathbb{R}_+ \subseteq \mathbb{R}$. Consider $F : X \rightrightarrows Y$ defined as

$$F(x) = [0, |x|], \text{ for all } x \in X.$$

Here, $\mathrm{Min}_L(A, F) = \{0\}$. Clearly, A and $\mathrm{Min}_L(A, F)$ are compact and hence all assumptions of Theorems 4.2 and 4.3 hold. We observe that $(P_{L,A})$ is LP well-posed in the set sense and

$$D(\epsilon) = \begin{cases} \{0\}, & \text{if } \epsilon = 0, \\ (-\epsilon, \epsilon), & \text{if } \epsilon > 0. \end{cases}$$

It can be seen that D is H-u.s.c. and closed at $\epsilon = 0$.

5 Stability aspects of $(P_{L,A})$

In this section, firstly we discuss the upper part of the Painlevé-Kuratowski convergence of a sequence of minimal solution sets of scalar problems which are defined by using generalized oriented distance function introduced in [27]. Secondly, we establish the upper part of the Painlevé-Kuratowski convergence of the solution sets of a sequence of perturbed set optimization problems where perturbations are obtained by perturbing the ordering cone and the feasible set.

Theorem 5.1 *Suppose that $(P_{L,A})$ is LP well-posed in the scalar sense and (v_n) is any sequence in $\mathrm{Min}_L(A, F)$ such that $v_n \to v \in \mathrm{Min}_L(A, F)$. Then,*

$$\mathrm{Ls}\ \mathrm{argmin}(A, \mathcal{D}_{F(v_n)-L} \circ F) \subseteq \mathrm{argmin}(A, \mathcal{D}_{F(v)-L} \circ F).$$

Proof Let $x \in \mathrm{Ls}\ \mathrm{argmin}(A, \mathcal{D}_{F(v_n)-L} \circ F)$. This implies that there exist $x_{n_k} \in \mathrm{argmin}(A, \mathcal{D}_{F(v_{n_k})-L} \circ F)$ such that $x_{n_k} \to x$ which further implies that $\mathcal{D}_{F(v_{n_k})-L}(F(x_{n_k})) = 0$. It follows that (x_{n_k}) is a LP minimizing sequence for $(P_{L,A})$ with respect to (v_{n_k}) in the scalar sense. Therefore, there exist subsequences $(v_{n_{k_l}})$ of (v_{n_k}) and $(x_{n_{k_l}})$ of (x_{n_k}) such that $v_{n_{k_l}} \to \hat{v}$ and $x_{n_{k_l}} \to \hat{x} \in \mathrm{argmin}(A, \mathcal{D}_{F(\hat{v})-L} \circ F)$. As $v_{n_k} \to v$ and $x_{n_k} \to x$, we obtain that $\hat{v} = v$ and $\hat{x} = x$. Thus, $x \in \mathrm{argmin}(A, \mathcal{D}_{F(v)-L} \circ F)$. Hence, $\mathrm{Ls}\ \mathrm{argmin}(A, \mathcal{D}_{F(v_n)-L} \circ F) \subseteq \mathrm{argmin}(A, \mathcal{D}_{F(v)-L} \circ F)$. □

Corollary 5.1 *Suppose that the problem $(P_{L,A})$ is LP well-posed in the scalar sense, $(v_n) \in \mathrm{Min}_L(A, F)$ is any sequence such that $v_n \to v \in \mathrm{Min}_L(A, F)$ and $F(v)$ is $(-L)$-proper and $(-L)$-closed. Then,*

$$\mathrm{Ls}\ \mathrm{argmin}(A, \mathcal{D}_{F(v_n)-L} \circ F) \subseteq \mathrm{Min}_L(A, F).$$

Remark 5.1 Theorem 5.1 generalizes Theorem 5.1 of [3] from vector case to the set case.

In the rest of the paper, the interior of the ordering cone L is assumed to be nonempty. We now investigate some stability aspects of the problem $(P_{L,A})$ by first perturbing the ordering cone L and then perturbing both the ordering cone L and the feasible set A.

We consider the following strict upper set-relation on $\mathcal{P}(Y)$ [17].

$$M \prec^u_L N \Leftrightarrow M \subseteq N - \mathrm{int} L, \text{ for } M, N \in \mathcal{P}(Y).$$

Kuroiwa [17] defined u-weak minimal solutions of the problem $(P_{L,A})$ using the above relation.

Definition 5.1 [17] An element $\hat{x} \in A$ is called a u-weak minimal solution of $(P_{L,A})$ if there does not exist any $x \in A$ such that $F(x) \prec^u_L F(\hat{x})$. The set of all u-weak minimal solutions of $(P_{L,A})$ are denoted by $\mathrm{WMin}_L(A, F)$.

We observe that $\mathrm{Min}_L(A, F) \subseteq \mathrm{WMin}_L(A, F)$ as pointed by Han and Huang [9]. Also, the reverse inclusion has been proved in [9] by using strictly upper L-convexity defined as follows:

Definition 5.2 [9] Let S be a nonempty convex set in X. The set-valued map F is called strictly upper L-convex on S if, for any $x_1, x_2 \in S$, $x_1 \neq x_2$ and for any $0 < \lambda < 1$, we have

$$F(\lambda x_1 + (1-\lambda)x_2) \subseteq \lambda F(x_1) + (1-\lambda)F(x_2) - \mathrm{int} L.$$

Lemma 5.1 [9] *Suppose that A is a convex set and F is strictly upper L-convex on the set A with nonempty convex compact values. Then* $\mathrm{Min}_L(A, F) = \mathrm{WMin}_L(A, F)$.

We next discuss the stability of $(P_{L,A})$ by considering the following sequence of perturbed problems where the ordering cone L is perturbed.

$$(P_{L_n,A}) \qquad \begin{aligned} &\min\nolimits_{L_n} \ F(x) \\ &\text{subject to} \quad x \in A \end{aligned}$$

where L_n is convex closed cone in Y with $\mathrm{int} L_n \neq \emptyset$. The sets of u minimal solutions and u-weak minimal solutions of $(P_{L_n,A})$ are denoted by $\mathrm{Min}_{L_n}(A, F)$ and $\mathrm{WMin}_{L_n}(A, F)$, respectively.

For the perturbed problems $(P_{L_n,A})$, we use the following form of Lemma 5.1.

Lemma 5.2 *Suppose that A is a convex set and F is strictly upper L_n-convex on the set A with nonempty convex compact values. Then* $\mathrm{Min}_{L_n}(A, F) = \mathrm{WMin}_{L_n}(A, F)$.

Theorem 5.2 *Suppose that (L_n) is a sequence of convex closed cone in Y such that* $\mathrm{Ls}\ (\mathrm{int} L_n)^c \subseteq (\mathrm{int} L)^c$*. If F is compact-valued and l.s.c. on A then* $\mathrm{Ls}\ \mathrm{WMin}_{L_n}(A, F) \subseteq \mathrm{WMin}_L(A, F)$.

Proof Let $\hat{x} \in \text{Ls}\,\text{WMin}_{L_n}(A, F)$. Then there exist $x_{n_k} \in \text{WMin}_{L_{n_k}}(A, F)$ such that $x_{n_k} \to \hat{x}$. If $\hat{x} \notin \text{WMin}_L(A, F)$ then there exists a point $x \in A$ such that $F(x) \prec^u_L F(\hat{x})$ which further implies that

$$F(x) \subseteq F(\hat{x}) - \text{int}L. \tag{6}$$

Since $x_{n_k} \in \text{WMin}_{L_{n_k}}(A, F)$, therefore $F(x) \nsubseteq F(x_{n_k}) - \text{int}L_{n_k}$ for all $k \in \mathbb{N}$ which further implies that there exist $y_{n_k} \in F(x)$ such that

$$y_{n_k} \notin F(x_{n_k}) - \text{int}L_{n_k}. \tag{7}$$

As $F(x)$ is compact, there exists a subsequence $(y_{n_{k_l}})$ of (y_{n_k}) such that $y_{n_{k_l}} \to y$ for some $y \in F(x)$. Using (6), we obtain that $y \in \hat{y} - \text{int}L$ for some $\hat{y} \in F(\hat{x})$ which in turn implies that

$$y - \hat{y} \in -\text{int}L. \tag{8}$$

As F is l.s.c. at $\hat{x}$, there exist $z_{n_{k_l}} \in F(x_{n_{k_l}})$ such that $z_{n_{k_l}} \to \hat{y}$. It follows from (7) that $y_{n_{k_l}} - z_{n_{k_l}} \in (-\text{int}L_{n_{k_l}})^c$ which means that $y - \hat{y} \in (-\text{int}L)^c$ which is a contradiction to (8). Therefore, $\hat{x} \in \text{WMin}_L(A, F)$ and hence $\text{Ls}\,\text{WMin}_{L_n}(A, F) \subseteq \text{WMin}_L(A, F)$. □

The following corollary can be proved on using Lemmas 5.1, 5.2 and Theorem 5.2.

Corollary 5.2 *Suppose that A is a convex set and (L_n) is a sequence of convex closed cone in Y such that* $\text{Ls}\,(\text{int}L_n)^c \subseteq (\text{int}L)^c$. *If F is compact-valued and* l.s.c. *on A, and F is strictly upper L-convex and L_n-convex on the set A with nonempty convex compact values then* $\text{Ls}\,\text{Min}_{L_n}(A, F) \subseteq \text{Min}_L(A, F)$.

The following example illustrates Theorem 5.2.

Example 5.1 Let $X = \mathbb{R}$, $A = [0, 1] \subseteq \mathbb{R}$, $Y = \mathbb{R}^2$, $L = \mathbb{R}^2_+ \subseteq \mathbb{R}^2$, $L_n = \left\{(x, y) : x - \frac{y}{2n} \geq 0 \text{ and } y - \frac{x}{2n} \geq 0\right\} \subseteq \mathbb{R}^2$. Consider $F : A \rightrightarrows Y$ defined as

$$F(x) = \begin{cases} [0, 1] \times [0, 1], & \text{if } x \neq 0, \\ \{(1, 1)\}, & \text{if } x = 0. \end{cases}$$

Here, all assumptions of Theorem 5.2 hold. We observe that $\text{WMin}_L(A, F) = [0, 1]$, $\text{WMin}_{L_n}(A, F) = [0, 1]$. Clearly, $\text{Ls}\,\text{WMin}_{L_n}(A, F) \subseteq \text{WMin}_L(A, F)$.

The next example illustrates that if F is not compact-valued then Theorem 5.2 may fails to hold.

Example 5.2 Let $X = \mathbb{R}$, $A = [0, 1] \subseteq \mathbb{R}$, $Y = \mathbb{R}^2$, $L = \mathbb{R}^2_+ \subseteq \mathbb{R}^2$, $L_n = \left\{(x, y) : x - \frac{y}{2n} \geq 0 \text{ and } y - \frac{x}{2n} \geq 0\right\} \subseteq \mathbb{R}^2$. Consider $F : A \rightrightarrows Y$ defined as

$$F(x) = \begin{cases} (0, 1) \times (0, 1), & \text{if } x \neq 0, \\ \{(1, 1)\}, & \text{if } x = 0. \end{cases}$$

Here, $\mathrm{WMin}_L(A, F) = \{0\}$, $\mathrm{WMin}_{L_n}(A, F) = [0, 1]$ and all assumptions of Theorem 5.2 hold but F is not compact-valued on A. Clearly, $1 \in \mathrm{Ls}\,\mathrm{WMin}_{L_n}(A, F)$ but $1 \notin \mathrm{WMin}_L(A, F)$. Hence, $\mathrm{Ls}\,\mathrm{WMin}_{L_n}(A, F) \nsubseteq \mathrm{WMin}_L(A, F)$.

In the last, we discuss the stability of $(P_{L,A})$ by considering the following sequence of perturbed problems where both the ordering cone L and the feasible set A are perturbed.

$$(P_{L_n,A_n}) \qquad \begin{array}{c} \min_{L_n} \; F(x) \\ \text{subject to} \quad x \in A_n \end{array}$$

where L_n is convex closed ordering cone in Y with $\mathrm{int}L_n \neq \emptyset$ and A_n is a nonempty set in X. The sets of u-weak minimal solutions and u-minimal solutions of (P_{L_n,A_n}) are denoted by $\mathrm{WMin}_{L_n}(A_n, F)$ and $\mathrm{Min}_{L_n}(A_n, F)$, respectively.

For the perturbed problems (P_{L_n,A_n}), Lemma 5.1 reduces to the following lemma.

Lemma 5.3 *Suppose that A_n is convex and F is strictly upper L_n-convex on A_n with nonempty convex compact values. Then* $\mathrm{Min}_{L_n}(A_n, F) = \mathrm{WMin}_{L_n}(A_n, F)$.

Theorem 5.3 *Suppose that (L_n) is a sequence of convex closed cone in Y such that* $\mathrm{Ls}\,(\mathrm{int}L_n)^c \subseteq (\mathrm{int}L)^c$, *$(A_n)$ is a sequence of nonempty sets in X such that* $A \subseteq \mathrm{Li}\,A_n$. *If F is compact-valued and* u.s.c *on A, and* l.s.c. *on X then* $\mathrm{Ls}\,\mathrm{WMin}_{L_n}(A_n, F) \subseteq \mathrm{WMin}_L(A, F)$.

Proof Let $\hat{x} \in \mathrm{Ls}\,\mathrm{WMin}_{L_n}(A_n, F)$. Then there exist $x_{n_k} \in \mathrm{WMin}_{L_{n_k}}(A_{n_k}, F)$ such that $x_{n_k} \to \hat{x}$. If $\hat{x} \notin \mathrm{WMin}_L(A, F)$ then there exists $x \in A$ such that $F(x) \subseteq F(\hat{x}) - \mathrm{int}L$. As $x \in A$, it follows that $x \in \mathrm{Li}\,A_n$ which further implies that there exist $w_n \in A_n$ such that $w_n \to x$. Since $x_{n_k} \in \mathrm{WMin}_{L_{n_k}}(A_{n_k}, F)$, therefore $F(w_{n_k}) \nsubseteq F(x_{n_k}) - \mathrm{int}L_{n_k}$ which means that there exist $y_{n_k} \in F(w_{n_k})$ such that $y_{n_k} \notin F(x_{n_k}) - \mathrm{int}L_{n_k}$. As F is u.s.c. at x, there exist $y \in F(x)$ and a subsequence $(y_{n_{k_l}})$ of (y_{n_k}) such that $y_{n_{k_l}} \to y$. Now, the proof follows on the lines of Theorem 5.2. □

By invoking Lemmas 5.1, 5.3 and Theorem 5.3, the following corollary can be easily proved.

Corollary 5.3 *Suppose that (A_n) is a sequence of convex sets in X such that* $A \subseteq \mathrm{Li}\,A_n$ *where $A \subseteq X$ is also convex and (L_n) is a sequence of convex closed cone in Y such that* $\mathrm{Ls}\,(\mathrm{int}L_n)^c \subseteq (\mathrm{int}L)^c$. *If F is compact-valued and* u.s.c. *on A,* l.s.c. *on X, strictly upper L-convex and L_n-convex on the set A and A_n, respectively with nonempty convex compact values then* $\mathrm{Ls}\,\mathrm{Min}_{L_n}(A_n, F) \subseteq \mathrm{Min}_L(A, F)$.

The following example illustrates Theorem 5.3.

Example 5.3 Let $X = \mathbb{R}$, $A = [0, 1] \subseteq \mathbb{R}$, $A_n = [0, 1 + 1/n] \subseteq \mathbb{R}$, $Y = \mathbb{R}^2$, $L = \mathbb{R}^2_+ \subseteq \mathbb{R}^2$, $L_n = \left\{(x, y) : x - \frac{y}{2n} \geq 0 \text{ and } y - \frac{x}{2n} \geq 0\right\} \subseteq \mathbb{R}^2$. Consider $F : X \rightrightarrows Y$ defined as

$$F(x) = \mathrm{Co}\,\{(0, 0), (x, (1 - x)), (x, x - 1)\}, \text{ for all } x \in X,$$

where Co S denotes the convex hull of a set S. Here, all assumptions of Theorem 5.2 are satisfied. It is observed that $\text{WMin}_L(A, F) = [0, 1]$, $\text{WMin}_{L_n}(A_n, F) = \left[0, 1 - \frac{1}{n}\right) \cup \left(1, 1 + \frac{1}{n}\right]$. Clearly, $\text{Ls WMin}_{L_n}(A_n, F) \subseteq \text{WMin}_L(A, F)$.

The next example justifies that Theorem 5.3 may not hold if F is not compact-valued.

Example 5.4 Let $X = \mathbb{R}$, $A = [0, 1] \subseteq \mathbb{R}$, $A_n = [-1/n, 1] \subseteq \mathbb{R}$, $Y = \mathbb{R}^2$, dom $F = (-\infty, 1]$, $L = \mathbb{R}^2_+ \subseteq \mathbb{R}^2$, $L_n = \left\{(x, y) : x - \frac{y}{2n} \geq 0 \text{ and } y - \frac{x}{2n} \geq 0\right\} \subseteq \mathbb{R}^2$. Consider $F : X \rightrightarrows Y$ defined as

$$F(x) = \begin{cases} \text{Co}\,\{(x, 0), (2, 1), (2, -2)\}, & \text{if } x < 0, \\ \text{int}(\text{Co}\,\{(0, 0), (x, 0), (x, x - 1)\}), & \text{if } 0 \leq x < 1, \\ [(0, 0), (1, 0)], & \text{if } x = 1. \end{cases}$$

We observe that $\text{WMin}_L(A, F) = \{1\}$, $\text{WMin}_{L_n}(A_n, F) = \{-1/n\}$. Clearly, F is not compact-valued on A, while all other assumptions in Theorem 5.2 hold. It can be seen that $0 \in \text{Ls WMin}_{L_n}(A_n, F)$ but $0 \notin \text{WMin}_L(A, F)$. Hence, the conclusion of Theorem 5.3 fails to hold.

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