Reconstruction of signals in quasi shift-invariant spaces from local averages

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***Abstract*—This article is concerned with the average sampling in quasi shift-invariant spaces $V_\alpha(\varphi)$ by considering the generator φ as a totally positive function (TPF) of finite type. A function φ is called the TPF of finite type N if $\widehat{\varphi}(w) = \prod_{l=1}^{N}(1+2\pi i\delta_l w)^{-1}$ for $0 \neq \delta_l \in \mathbb{R}$ and $N \geq 2$. We prove that if sampling points are close enough, then all signals belonging to the quasi shift-invariant space are reconstructed stably and uniquely by using its average sample values. An efficient iterative frame reconstruction algorithm for reconstruction of a signal $g \in V_\alpha(\varphi)$ by using its average sample values is also provided.**

***Index Terms*—Nonuniform sampling, average sampling, quasi shift-invariant space, TPFs of finite type, iterative reconstruction algorithm**

I. Introduction

Sampling and reconstruction play a significant role in several research areas like signal and modern digital data processing. Sampling problem mainly consists of two parts. The first part is to find conditions on a sampling set $Y = \{y_k : k \in \mathbb{Z}\}$ such that any g belonging to a class of functions (signals) V can be recovered in a stable and unique way by using its sample values $\{g(y_k) : k \in \mathbb{Z}\}$ and the other part involves providing an efficient numerical algorithm for recovery of a signal $g \in V$ by using its sample values.

The celebrated Shannon sampling theorem tells that every signal belonging to the shift-invariant space $V(sinc)$ can be recovered in stable and unique way by using its sample values $\{g(k) : k \in \mathbb{Z}\}$ as follows

$$g(y) = \sum_{k\in\mathbb{Z}} g(k)\frac{\sin\pi(y-k)}{\pi(x-k)}, \qquad y \in \mathbb{R}.$$

However, many practical situations do not have uniform sampling set in general. A sampling set $Y = \{y_k : k \in \mathbb{Z}\}$ is called separated if $\inf_{k\in\mathbb{Z}}(y_{k+1} - y_k) = d > 0$, where d is the separation of Y. For band-limited signals with finite energy, Gröchenig [1] studied nonuniform sampling problem and proved that if $Y = \{y_k : k \in \mathbb{Z}\}$ is any separated subset of $\mathbb{R}$ satisfying $\sup_{k\in\mathbb{Z}}(y_{k+1} - y_k) < 1$, then any $g \in V(sinc)$ can be recovered stably and uniquely by taking its samples $\{g(y_k) : k \in \mathbb{Z}\}$. Moreover, Gröchenig [1] also provided an iterative approximation-projection algorithm with exponential convergence for reconstruction of signals belonging to $V(sinc)$.

Aldroubi and Gröchenig [2] examined the sampling problem in B-spline spaces. Recall that a B-spline signal of order n is defined inductively as

$$Q_1 = \chi_{[0,1]} \quad \text{and} \quad Q_n = Q_{n-1} * Q_1, \quad n \geq 2.$$

Aldroubi and Gröchenig [2] proved that for any separated subset Y of $\mathbb{R}$ satisfying $\sup_{k\in\mathbb{Z}}(y_{k+1} - y_k) < 1$, any $g \in V(Q_n)$ can be recovered stably and uniquely by using its nonuniform samples $\{g(y_k) : k \in \mathbb{Z}\}$. Kulkarni et al. [3] discussed the sampling problem in shift-invariant space $V(\varphi)$ and proved that with certain mild assumptions on φ, every $g \in V(\varphi)$ can be reconstructed stably and uniquely by using its samples taken on a sampling set obtained by perturbation of integers. Recently, Selvan and Radha [4] studied the nonuniform sampling for a scaling function φ corresponding to the Meyer wavelet and proved that for any separated subset Y of $\mathbb{R}$ satisfying $\sup_{k\in\mathbb{Z}}(y_{k+1} - y_k) < 1$, sample values $\{g(y_k) : k \in \mathbb{Z}\}$ are used to reconstruct any signal $g \in V(\varphi)$ stably and uniquely. For more details about sampling problem in $V(\varphi)$, we refer [5]–[9] and the references therein.

Due to physical reasons, it is not always easy to capture the exact signal value $g(y_k)$ of g at a particular instance y_k. Instead of measuring the exact value of the signal particularly at time y_k, only averages can be measured in practice. Specifically, measured average sample values are taken as

$$\langle g, u_k\rangle = \int_{\mathbb{R}} g(x)u_k(x)dx.$$

Here, the collection $\{u_k : k \in \mathbb{Z}\}$ of average functions satisfies the conditions

$$\text{supp } u_k \subset [y_k - \tfrac{\delta}{2}, y_k + \tfrac{\delta}{2}], \quad \int_{\mathbb{R}} u_k(x)dx = 1 \quad \text{and} \quad u_k \geq 0.$$

If δ is small enough, then it can be easily seen that the exact signal can be well approximated by taking its local averages. In the average sampling problem, we are interested to find the range of δ such that any g belonging to a class of functions V can be reconstructed stably and uniquely by using its average sample values taken on discrete sampling sets. For Ω band-limited signals with finite energy, Sun and Zhou [10] proved that for any separated subset Y of $\mathbb{R}$ satisfying $\sup_{k\in\mathbb{Z}}(y_{k+1} - y_k) = \beta < \frac{\pi}{\Omega}$ and $0 < \delta < \frac{\pi}{\Omega} - \beta$, any $g \in PW_\Omega$ can be reconstructed stably and uniquely by

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